LECTURE

6

HIGHER ORDER DERIVATIVES. TAYLOR SERIES AND POWER SERIES

Higher order derivatives

Definition 6.1 Let A ⊆ R, c ∈ A ∩ A and f : A → R. We say that f is twice differentiable at c if ∃V ∈ V(c) such that f is differentiable on A ∩ V and f is differentiable at c. If f is twice differentiable at c, then we write

f (2)(c) .= .f (c) .= .(f ) (c).

In general, for n ∈ N, ≥ 2, we say that f is n-times differentiable at c if ∃V ∈ V(c) such that f is (n−1)-times differentiable on A∩V and f(n−1) is differentiable at c. If f is n-times differentiable at c, then we write

f(n)(c) ..= (f (n−1)) (c). If B is a nonempty subset of A, we say that f is n-times differentiable on B if it is n-times differentiable at every point of B. In this case, the function f(n) : B → R, x ∈ B ↦→ f (n)(x) ∈ R is called the nth derivative of f on B.

We say that f is infinitely differentiable at c if f is n-times differentiable at c for every n ∈ N. Notational conventions: f(0) .= .f and f (1) .= .f .

Approximation of differentiable functions by Taylor polynomials

Let I ⊆ R be an interval, x0 ∈ I, f : I → R and n ∈ N. Supposing that f is n-times differentiable at x0, we want to find a polynomial function P : R → R, of degree (at most) n, such that 

P(x0) = f(x0) P (x0) = f (x0) P (x0) = ... f (x0) P (n)(x0) = f(n)(x0).

(6.1)

We are looking for P of the form

P(x) = a0 + a1(x − x0) + a2(x − x0)2 + ... + an(x − x0)n.

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By (6.1) we deduce that

a0 = f(x0), a1 = f (x0), a2 = f 2! (x0)

, ..., an = f(n)n! (x0)

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So, there is a unique polynomial P of degree (at most) n satisfying (6.1).

Definition 6.2 Let I ⊆ R be an interval, x0 ∈ I, f : I → R and n ∈ N. Supposing that f is n-times differentiable at x0, the polynomial function Tn : R → R, given by

Tn(x) .= .f(x0) + f (x1! 0)

(x − x0) + ... + f(n)n! (x0)

(x − x0)n (6.2)

is called the nth Taylor polynomial of f (centered) at x0.

Remark 6.3 The nth Taylor polynomial of f at x0 is also denoted by Tn(f;x0). However, we simply write Tn(x) instead of Tn(f;x0)(x) for all x ∈ R.

Remark 6.4 Since the Taylor polynomial satisfies

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Tn(x0) = f(x0) T T nn(x(x0) 0) = f (x0) = ... f (x0) T n (n) (x0) = f(n)(x0),

(6.3)

it approximates the function f on a neighborhood of x0, i.e.,

f(x) ≃ f(x0) + f (x1! 0)

(x − x0) + ... + f(n)n! (x0)

(x − x0)n.

In particular, for n = 1 we obtain

f(x) ≃ f(x0) + f (x0)(x − x0).

Definition 6.5 Let I ⊆ R be an interval, x0 ∈ I, f : I → R and n ∈ N. Supposing that f is n-times differentiable at x0, the function Rn : I → R, defined by

Rn(x) ..= f(x) − Tn(x), ∀x ∈ I, (6.4)

is called the remainder of the approximation of f by Tn around x0. Whenever Rn is given explicitly, we get the so-called Taylor formula:

f(x) = f(x} 0) + f (x1! 0)

(x − x0) {{ + ... + f (n)n! (x0)

(x − x0)n } Tn(x)

+Rn(x), ∀x ∈ I.

Theorem 6.6 (Taylor-Lagrange) Let f : I → R be a function which is (n+1)-times differentiable on I for some n ∈ N ∪ {0}. Then, for any distinct points x, x0 ∈ I there exists a point c ∈ R, min{x0,x} <c< max{x0,x}, such that

f(x) = f(x0) + f (x1! 0)

(x − x0) + ... + f(n)n! (x0)

(x − x0)n + f(n (n+1)+ 1)! (c)

(x − x0)n+1. (6.5)

In other words, we have f(x) = Tn(x) + Rn(x), where

Rn(x) = f(n (n+1)+ 1)! (c)

(x − x0)n+1. (6.6)

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Proof. Consider any distinct points x, x0 ∈ I. Without loss of generality we can assume that x0 < x. By (6.3) and (6.4) we have

R(k) n (x0)=0, ∀k ∈ {0,1,...,n}. By Cauchy’s Generalized Mean Value Theorem 5.48, applied to the functions

x ↦− → Rn(x) and x ↦− → (x − x0)n+1

on the interval [x0,x], there exists c1 ∈ (x0,x) such that

Rn(x) (x − x0)n+1 = Rn(x) − Rn(x0)

(x − x0)n+1 − (x0 − x0)n+1 = R n(c1)

(n + 1)(c1 − x0)n.

Applying now Cauchy’s Generalized Mean Value Theorem to the functions

x ↦− → R n(x) and x ↦− → (n + 1)(x − x0)n

on the interval [x0,c1], we deduce that there is ∃c2 ∈ (x0,c1) such that

Rn(x) (x − x0)n+1 = R n(c1)

(n + 1)(c1 − x0)n = R n(c1) − R n(x0)

(n + 1)(c1 − x0)n − (n + 1)(x0 − x0)n = R n(c2)

(n + 1)n(c2 − x0)n−1.

Continuing in this way we find cn+1 ∈ (x0,cn) ⊆ (x0,cn−1) ⊆···⊆ (x0,x) such that

Rn(x) (x − x0)n+1 = R(n+1) n (cn+1)

(n + 1)! . (6.7)

On the other hand, recalling that Tn is a polynomial of degree at most n, we deduce by (6.4) that R(n+1) n (cn+1) = fn+1(cn+1) − T (n+1) n (cn+1) = f n+1(cn+1). Hence, by choosing c ..= cn+1, we infer from (6.7) that

Rn(x) = f(n+1)(c)

(n + 1)! (x − x0)n+1. D

Remark 6.7 (6.5) is called the Taylor’s formula with the remainder in Lagrange’s form (6.6).

Remark 6.8 Assume that, for some a, b ∈ I with a < x0 < b, there exists M ∈ R such that |f(n+1)(x)| ≤ M for all x ∈ [a, b]. Then, the error of approximation of f(x) by Tn(x) can be estimated by

|f(x) − Tn(x)| ≤ M

(n + 1)!(x − x0)n+1, ∀x ∈ [a, b].

Corollary 6.9 (Local optimality conditions) Let f : I → R be a function, defined on an interval I ⊆ R. If f is n-times differentiable (n ∈ N, n ≥ 2) at x0 ∈ int,I and

f (x0) = f (x0) = ··· = f(n−1)(x0)=0 = f (n)(x0), then the following assertions hold true: 1◦ If n is even and f (n)(x0) > 0, then x0 is a local minimum point of f. 2◦ If n is even and f (n)(x0) < 0, then x0 is a local maximum point of f. 3◦ If n is odd, then x0 is not a local extremum point of f.

Example 6.10 Let f : R → R, defined by f(x) = 24 cosx + 12x2 − x4 for all x ∈ R. It is easy to check that

f (0) = f (0) = f (3)(0) = f(4)(0) = f (5)(0) = 0 = −24 = f(6)(0),

hence x0 = 0 is local maximum point of f.

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Taylor series

Definition 6.11 Let I ⊆ R be an interval and let f : I → R be infinitely differentiable. For x0 ∈ I and x ∈ R, the series ∑n≥0

f(n)n! (x0)

(x − x0)n (6.8)

is called the Taylor series of f around x0.

If J ⊆ I is a nonempty set such that for every x ∈ J the series (6.8) converges and its sum is f(x), i.e.,

f(x) =

∑∞n=0

f (n)n! (x0)

(x − x0)n, (6.9)

then we say that f can be expanded as a Taylor series around x0 on J. In this case, (6.9) is called the Taylor expansion of f(x) around x0 on J.

Remark 6.12 For any x ∈ I, the partial sums of the Taylor series (6.8) are given by

∑nk=0

f(k)k! (x0)

(x − x0)k = Tn(x), ∀n ∈ N ∪ {0}.

Thus, the series (6.8) converges if and only if its sum is finite, i.e.,

∑∞n=0

f(n)n! (x0)

(x − x0)n .= .s(x) .= .n→+∞lim Tn(x) ∈ R

and, according to (6.4), we have

n→+∞lim Rn(x) = f(x) − n→+∞lim Tn(x) = f(x) − s(x).

Therefore, by Definition 6.11, f can be expanded as a Taylor series around x0 on J if and only if

n→+∞lim Rn(x)=0, ∀x ∈ J.

Example 6.13 (Taylor expansion of the exponential function around 0) Let f : R → R,

f(x) = ex.

Note that ∀k ∈ N, ∀x ∈ R, f (k)(x) = ex, so ∀k ∈ N, f (k)(0) = 1. Let n ∈ N, x ∈ R. Then there exists c between 0 and x such thatex =1+ 1!1x + ... + n!1xn + Rn(x),

where Rn(x) = (n + ec

1)!xn+1. Since 0 ≤ |Rn(x)| ≤ ec (n |x|+ n+1

1)! and n→∞

lim |x|n! n= 0, it follows that n→∞lim Rn(x)=0. Therefore, f can be expanded as a Taylor series around 0 on R:

ex =

∑∞n=0

xn!n, ∀x ∈ R.

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Example 6.14 (Taylor expansion of sine function around 0) The function f : R → R,

f(x) = sinx,

can be expanded as a Taylor series around 0 on R:

sinx =

∑∞n=1(−1)n−1 (2n x2n−1

− 1)!, ∀x ∈ R.

Example 6.15 (Taylor expansion of cosine function around 0) The function f : R → R,

f(x) = cosx,

can be expanded as a Taylor series around 0 on R:

cosx =

∑∞n=1(−1)n−1 (2n x2n−2

− 2)!, ∀x ∈ R.

Power series

Definition 6.16 Let (an)n≥0 be a sequence of real numbers and let c ∈ R. A series of type

∑n≥0

an(x − c)n, where x ∈ R, (6.10)

is called power series centered at x with coefficients an. The set

C ..= {x ∈ R | the series (6.10) converges}

is called the convergence set of the power series.

Theorem 6.17 (Abel) There exists R ∈ [0,+∞) ∪ {+∞} such that the power series (6.10) con- verges absolutely whenever 0 ≤ |x − c| < R and diverges whenever |x − c| > R.

Definition 6.18 The unique R satisfying the conditions of Abel’s Theorem 6.17 is called the radius of convergence of the power series.

Theorem 6.19 (Cauchy-Hadamard) If the limit

L := n→∞

lim √n|an| ∈ [0,∞) ∪ {+∞}

exists, then the power series (6.10) has the radius of convergence R =

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1/L if 0 <L< +∞

0 if L = +∞ +∞ if L = 0.

Corollary 6.20 If the limit

L := n→∞lim ∣∣∣∣an+1

an

∣∣∣∣ ∈ [0,∞) ∪ {+∞}

exists, then the power series (6.10) has the radius of convergence R =

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1/L if 0 <L< +∞

0 if L = +∞ +∞ if L = 0.

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Example 6.21 1) For the geometric series ∑n≥1(x − c)n, centered at any number c ∈ R, we have R = 2) 1 For and ∑C = (c − 1,c + 1).

n≥1

n1xn we have R = 1 and C = [−1,1). 3) For ∑n≥1

(−1)n n

xn we have R = 1 and C = (−1,1].

4) 5) For For ∑∑n≥1

n!1(x − c)n, centered at any c ∈ R, we have R = +∞ and C = R.

n≥1

n!(x + 1)n we have R = 0 and C = {−1}.

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